Challenge Question (from 2008 Putnam Competition)

We start with an "empty" matrix P that is 2008×2008

John and Mary play the following game:

> John picks any number and enters it somewhere in the matrix.
>
> Then Mary picks a number and puts it somewhere in the matrix
>
> Then John enters another number, then Mary, etc., taking turns back and forth until the matrix is filled.
>
> The det G is computed. John wins the game if $\det G \neq 0$; Mary wins if $\det G = 0$.

Does either payer have a winning strategy $-$ that is, an algorithm for making plays that will guarantee a win?

Review from last time:

Theorem Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation, with $T(\boldsymbol{x}) = A\boldsymbol{x}$

1) if S is any subset of $\mathbb{R}^2$ that has a finite area, then

$$\text{area } (T(S)) = |\det A| \cdot \text{area}(S)$$

2) if S is any subset of $\mathbb{R}^3$ that has a finite volume, then

$$\text{volume } (T(S)) = |\det A| \cdot \text{volume}(S)$$

3) *If "volume" is generalized to* $\mathbb{R}^n$ $(n > 3)$*, the same results is true for "solids" in* $\mathbb{R}^n$

if S is any subset of the $\mathbb{R}^n$ that has a finite volume, then

$$\text{volume } (T(S)) = |\det A| \cdot \text{volume}(S)$$

Quick presentation in class of how this fact is reflected in the formula for making substitutions in a double integral

We've been talking about $\mathbb{R}^n$ since the beginning of the course. How do vectors in $\mathbb{R}^n$ behave? The crucial properties:

The vector space $\mathbb{R}^n$ consists of vectors like $\boldsymbol{u} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$. There are two operations on vectors:

i) "addition" $(+)$
ii) "multiplication by a scalar" such that:

They obey the following rules:

For all $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in V$ (vectors)
and for all $c, d \in \mathbb{R}$ (scalars)

1. $u + v$ is in V (closure under addition)
2. $u + v = v + u$
3. $(u + v) + w = u + (v + w)$
4. There is a vector (denoted 0) with the property $u + 0 = u$
5. For each vector u, there is another vector called $-u$ for which $u + (-u) = 0$

6. cu is in V (closure under scalar multiplication)
7. $c(u + v) = cu + cv$
8. $(c + d)u = cu + du$
9. $c(du) = (cd)u$
10. $1u = u$

Now we generalize: any collection of objects that "behaves like the vector space $\mathbb{R}^n$ will be called a vector space, and its member will be called vectors:

(which we will call vectors)

Definition: A vector space V is a nonempty set of objects ↕ and two operations which we call

i) "addition" $(+)$
ii) "multiplication by a scalar" such that:

For all $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in V$ (vectors)
and for all $c, d \in \mathbb{R}$ (scalars)

1. $u + v$ is in V (closure under addition)
2. $u + v = v + u$
3. $(u + v) + w = u + (v + w)$
4. There is a vector (denoted 0) with the property $u + 0 = u$
5. For each vector u, there is another vector called $-u$ for which $u + (-u) = 0$

6. $c\boldsymbol{u}$ is in V (closure under scalar multiplication)
7. $c(\boldsymbol{u} + \boldsymbol{v}) = c\boldsymbol{u} + c\boldsymbol{v}$
8. $(c + d)\boldsymbol{u} = c\boldsymbol{u} + d\boldsymbol{u}$
9. $c(d\boldsymbol{u}) = (cd)\boldsymbol{u}$
10. $1\boldsymbol{u} = \boldsymbol{u}$

From these axioms for a vector space A1-A10, other properties follow such as:

for every vector $\boldsymbol{u}$, $0 \cdot \boldsymbol{u} = \mathbf{0}$

because

$$0 \cdot \boldsymbol{u} = (0+0) \cdot \boldsymbol{u} \; = 0 \cdot \boldsymbol{u} + 0 \cdot \boldsymbol{u}$$

Add $-(0 \cdot \boldsymbol{u})$ to both sides

$$0 \cdot \boldsymbol{u} - (0 \cdot \boldsymbol{u}) = (0 \cdot \boldsymbol{u} + 0 \cdot \boldsymbol{u}) - (0 \cdot \boldsymbol{u})$$

$$\mathbf{0} \qquad = 0 \cdot \boldsymbol{u} + (0 \cdot \boldsymbol{u} - (0 \cdot \boldsymbol{u}))$$

$$\mathbf{0} \qquad = 0 \cdot \boldsymbol{u} + \mathbf{0} = 0 \cdot \boldsymbol{u}$$

and so

$$\mathbf{0} = 0 \cdot \boldsymbol{u}$$

Examples of vector spaces:

$\mathbb{P}$ = the collection of all polynomials

$\mathbb{P}_2$ = the collection of all polynomials with degree ≤ 2

Definition: Suppose $V =$ a vector space
and that H is a subset of V (written $H \subseteq V$)

H is called a subspace of V if

i) $\mathbf{0} \in H$

ii) if $\boldsymbol{u}, \boldsymbol{v} \in H$, then $\boldsymbol{u} + \boldsymbol{v} \in H$
(H is "closed under addition")

iii) if $\boldsymbol{u} \in H$ and c is a scalar, then $c\boldsymbol{u} \in H$
(H is "closed under scalar multiplication")

All other properties needed for a vector space are satisfied by vectors in H automatically because they are true already in the given vector space V:

for example, if $\boldsymbol{u}, \boldsymbol{v} \in H$, then $\boldsymbol{u} + \boldsymbol{v} = \boldsymbol{v} + \boldsymbol{u}$
because $\boldsymbol{u}, \boldsymbol{v} \in V$ we already know
$\boldsymbol{u} + \boldsymbol{v} = \boldsymbol{v} + \boldsymbol{u}$ is true in V.

Think of a subspace H as a smaller "self-contained" vector space that lives inside the larger vector space V.

Example: Let $1, t, t^2$ be e vectors in $\mathbb{P}$. Then $H =$ Span $\{1, t, t^2\}$ = the collection of all linear combinations of these vectors $= \{c_0 + c_1 t + c_2 t^2 : c_0, c_1, c_2 \text{ scalars}\} = \mathbb{P}_2$, a subspace of $\mathbb{P}$
because i) $\mathbf{0} = 0 + 0t + 0t^2$ is in H.
ii) if $\boldsymbol{u}$ and $\boldsymbol{v}$ are in $\mathbb{P}_2$, so is $\mathbf{u} + v$
iii) uf $\boldsymbol{u}$ is in $\mathbb{P}_2$, so is $c\boldsymbol{u}$

In general: For $\boldsymbol{v_1}, \boldsymbol{v_2}, ..., \boldsymbol{v_p}$ are vectors in V,

let $H =$ **span** $\{v_1, v_2, ..., v_p\}$

$=$ {all possible linear combinations of $v_1, v_2, ..., v_p$}

Then H is a subspace of V

Why?

i) $\mathbf{0} = 0v_1 + ... + 0v_p$ is in H

ii) If $u, v \in H$, that means

$u = c_1v_1 + ... + c_pv_p$ for some weights $c_1, ..., c_p$ and

$v = d_1v_1 + ... + d_pv_p$ for some weights $d_1, ..., d_p$

Then $u + v = (c_1 + d_1)v_1 + ... + (c_p + d_p)v_p$

so $u + v$ is in H, and

iii) $cu = (cc_1)\, v_1 + ... + (cc_p)\, v_p$ is in H

Example: The set of all polynomials with degree > 2 is not a subspace of $\mathbb{P}$.
For example, the sum of two polynomials each with degree > 2 might not have degree > 2 : for example $u(t) = t^3$ and $v(t) = -t^3$

Example : A plane H in $\mathbb{R}^3$ that contains the origin is a subspace, because if we pick two linearly independent vectors in H, then $H = \text{Span}\{u, v\}$

If p is a vector not in H and we look at the translation of the plane:

$p + H$

this set of vectors is not a subspace, because (just for example) $\mathbf{0}$ is not in the plane $p + H$.