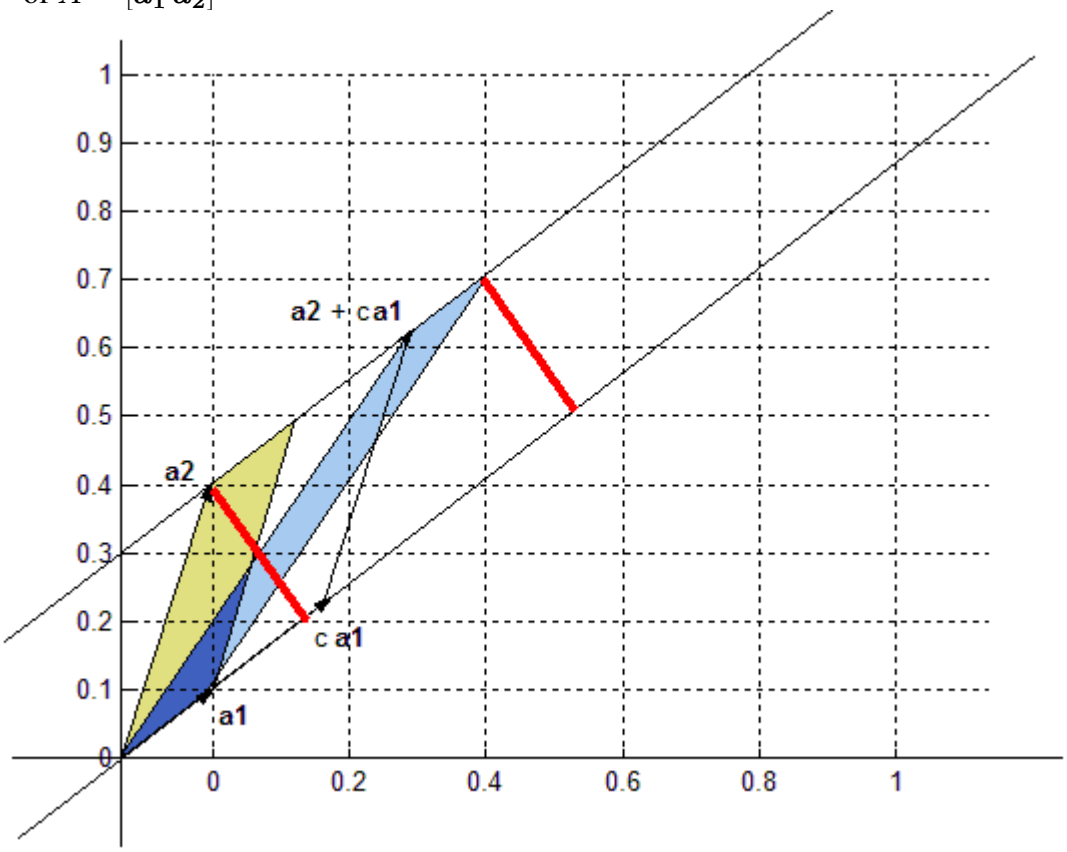
(*These pdfs supplement the text and lecture; you can't just read them and understand the lecture; lots of things aren't in the pdfs)*

Column replacement doesn't change the area of the parallelogram determined by the columns of $A = [\boldsymbol{a_1}\ \boldsymbol{a_2}]$



Area of parallelogram determined by $\boldsymbol{a_1}$ and $\boldsymbol{a_2}$
(*brown and dark blue*)

$=$ Area of parallelogram determined by $\boldsymbol{a_1}$ and $\boldsymbol{a_2} + c\boldsymbol{a_1}$
(*blue and dark blue*)

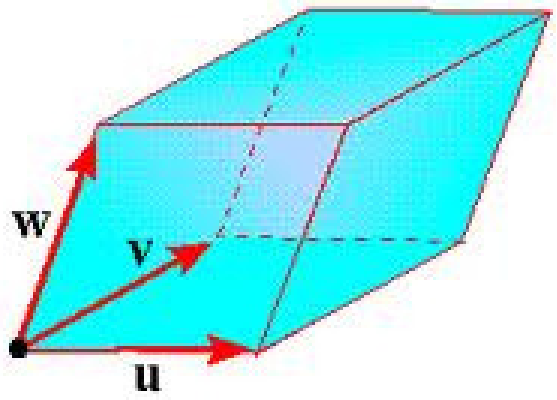
because they have

the same base b = length of vector $\boldsymbol{a_1}$ and
the same height h = length of red line segment

Lecture: Area of parallelogram determined by $\boldsymbol{a_1}, \boldsymbol{a_2}$ $=$ $= |\det A|$

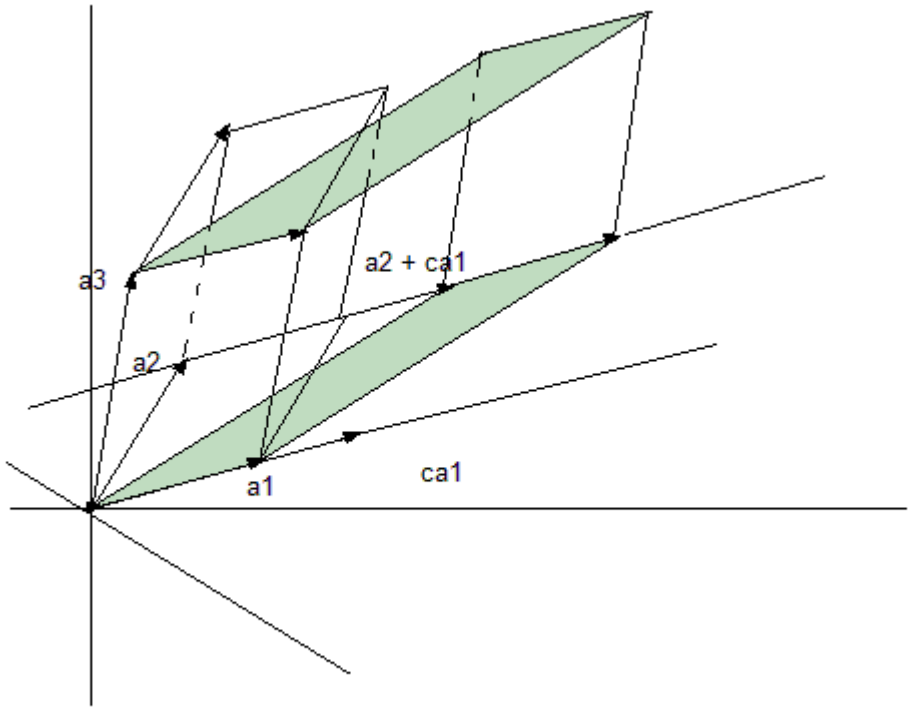
***The same is true for <u>volumes</u>* in $\mathbb{R}^3$**

For a 3×3 matrix $A = [\boldsymbol{u}\ \boldsymbol{v}\ \boldsymbol{w}]$



Volume of the parallelepiped determined by $u, v, w = |$ ***det*** $[u\ v\ w]|$

Volume of parallelepiped not affected by a column replacement operation.

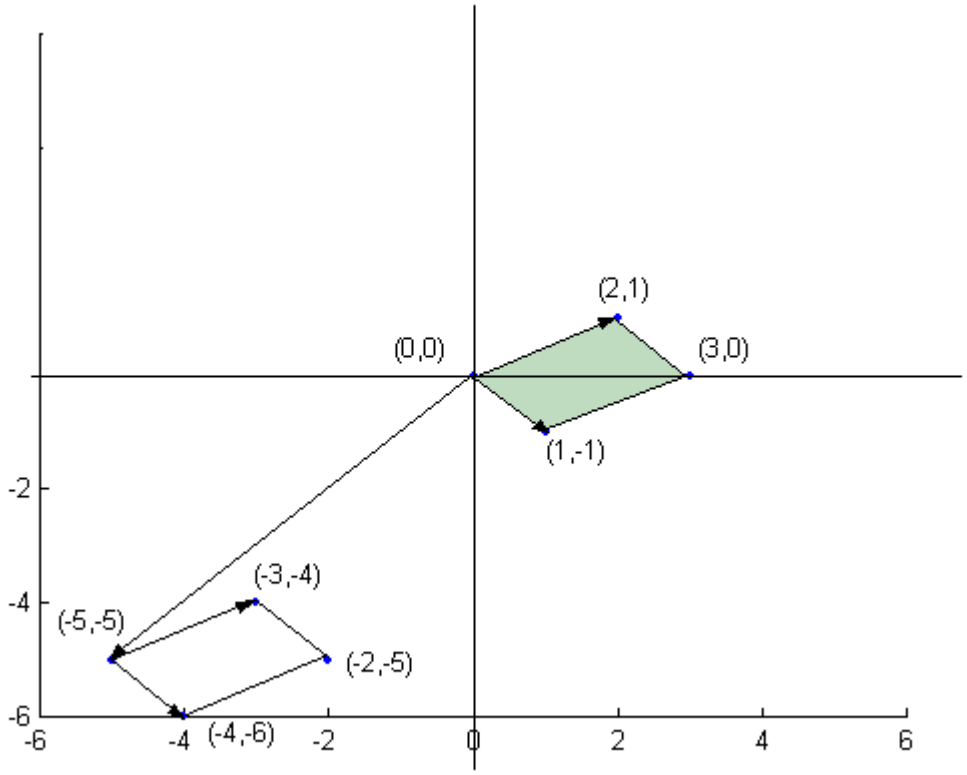


In higher dimensions e.g., $\mathbb{R}^4$,

we can <u>define</u> volume so that if A is 4×4, then "volume" of "parallelepiped" determined by columns of A is $|\det A|$

Find area of parallelogram with vertices

$$\begin{bmatrix} -5 \\ -5 \end{bmatrix}, \quad \begin{bmatrix} -4 \\ -6 \end{bmatrix}, \quad \begin{bmatrix} -3 \\ -4 \end{bmatrix}, \quad \begin{bmatrix} -2 \\ -5 \end{bmatrix}$$



Translate one vertex to **0**, e.g., add $\begin{bmatrix} 5 \\ 5 \end{bmatrix}$ to all vertices.

Parallelograms have same area: Area $= |\det \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}| = 3$

Area of $\triangle$ with vertices

$\begin{bmatrix} -5 \\ -5 \end{bmatrix}, \quad \begin{bmatrix} -4 \\ -6 \end{bmatrix}, \quad \begin{bmatrix} -3 \\ -4 \end{bmatrix}$: Area $= \frac{1}{2}$(area of parallelogram) $= \frac{3}{2}$

Effect of a Linear Transformation on Area

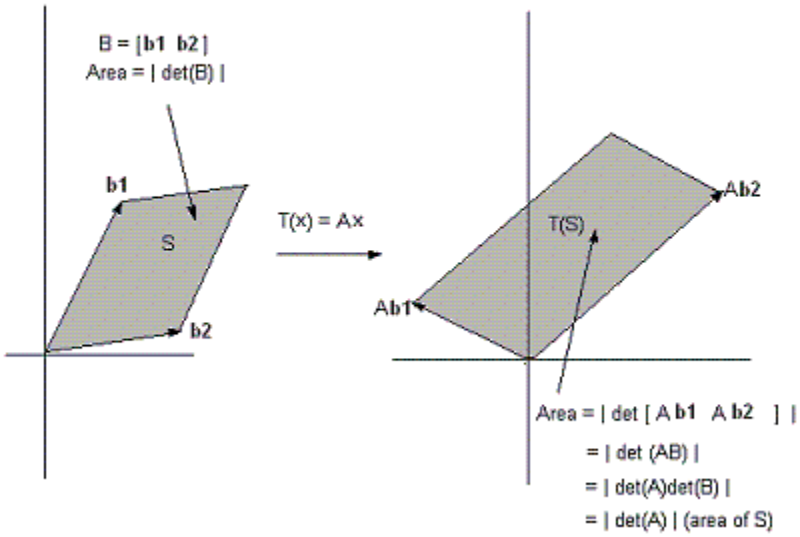
Theorem Suppose $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation with standard matrix A.

1) Let S be a parallelogram in $\mathbb{R}^2$. The image of the parallelogram is $T(S) = \{T(\boldsymbol{x}) : \boldsymbol{x} \text{ in } S\}$ and

↑

another parallelogram (possibly "collapsed" to line segment or point)

$$(\text{area of } T(S)) = |\det A| \cdot (\text{area of } S)$$



2) If $T : \mathbb{R}^3 \to \mathbb{R}^3$ is linear with standard matrix A. Let S be a parallelepiped in $\mathbb{R}^3$. Then

$$(\text{volume of } T(S)) = |\det A| \cdot (\text{volume of } S)$$

And not just for parallelograms!

"Theorem" For linear $T : \mathbb{R}^2 \to \mathbb{R}^2$, or for $T : \mathbb{R}^3 \to \mathbb{R}^3$

1) if S is any subset of the plane that has a finite area, then

$$\text{area } (T(S)) = |\det A| \cdot \text{area}(S)$$

2) if S is any subset of the $\mathbb{R}^3$ that has a finite volume, then

$$\text{volume } (T(S)) = |\det A| \cdot \text{volume}(S)$$