EFFECT OF EROs ON DETERMINANTS

Let A be a square matrix:

1) if a multiple of one row of A is added to another to get a matrix B, then $\det A = \det B$
(*row replacement has no effect on determinant*)

2) If two rows of A are interchanged to get B, then $\det B = -\det A$
(*each row swap reverses the sign of the determinant*)

3) If one row of A is multiplied by k ($\neq 0$) to get B, then $\det B = k \det A$
(*rescaling a row by a factor of k also rescales the determinant by a factor of k)*

Example: $\det \underset{B}{\begin{bmatrix} 5a & 5b \\ c & d \end{bmatrix}} = 5 \det \underset{A}{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}$

(3) often used to "factor out" a number from a row

Why?

Rephrase in terms of elementary matrices

Theorem Suppose A is $n \times n$ and
E is an $n \times n$ elementary matrix

then 1. $\det(EA) = \det(E)\det(A)$

and ↗

2. $\det(E) =$

$$\begin{cases} 1 & \text{if } E \text{ is: add multiple of row to another} \\ -1 & \text{if } E \text{ is: interchange two rows} \\ k & \text{if } E \text{ is: rescale a row by a factor of } k \end{cases}$$

_____ WHY?_____

Check this first for 2×2 matrices

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is 2×2 and that E is a 2×2 elementary matrix ($\leftrightarrow$ ERO)

1. If E represents a row replacement (say, k*row 1 added to row 2)

$$EA = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix} \cdot A = \begin{bmatrix} a & b \\ ka + c & kb + d \end{bmatrix},$$

$\det(E) = 1$

so

$\det(EA) = kab + ad - kab - bc = ad - bc$

$= \det(A)$ (*so this ERO doesn't change* $det(A)$)

$= \det(E) \cdot \det(A)$
↑
1

2. If E represents a row rescaling (say, a rescaling of row 1)

$$EA = \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix} \cdot A = \begin{bmatrix} ka & kb \\ c & d \end{bmatrix}$$

$\mathbf{\det(E) = k}$

so

$\mathbf{\det(EA)} = k(ad - bc) \quad = k\det(A)$

(*so this ERO multiplies* $det(A)$ *by* k)

$= \mathbf{\det(E)} \cdot \mathbf{\det(A)}$
$\uparrow$
k

3. If E represents a row interchange

$$EA = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot A = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$$

$\mathbf{\det(E) = -1}$

so

$\mathbf{\det(EA)} = (bc - ad) = (-1)\det(A)$

(*so this ERO changes sign of det* A)

$= \mathbf{\det(E)} \cdot \mathbf{\det(A)}$
$\uparrow$
-1

<u>So, by direct checking we find that the Theorem is true for 2×2 matrices.</u>

Next argue
Theorem true for 2×2 matrices:
forces the theorem to be true for 3×3 matrices;

Then Theorem true for 3×3 matrices:
forces the theorem to be true for 4×4 matrices;

And in general

$\vdots$

Theorem true for $n \times n$ matrices
forces the theorem to be true for $(n+1) \times (n+1)$ matrices.

This is called a proof by induction: we argue "up the ladder": it's true for 2×2 matrices, and whenever it's true for one size, $n \times n$, then it must also be true for the "next size up", $(n+1) \times (n+1)$. This means it must be true for all sizes of square matrices.

We didn't do this general argument. But illustrated by showing why, for a row replacement ERO, "true for 2×2" forces "true for 3×3". The same style argument is how you argue from size $n \times n$ to size $(n+1) \times (n+1)$.

To illustrate: Why does the fact that

$$(**) \qquad \det(EA) = \det(E)\det(A);$$

works for 2×2 force it also to be true for 3×3 ?

For Row replacement:

$$\det(E) = 1 \quad \text{(E)} \qquad \text{(A)} \qquad \text{(EA)}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ k & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} a & b & c \\ d+ka & e+kb & f+kc \\ \boldsymbol{g} & \boldsymbol{h} & \boldsymbol{i} \end{bmatrix}$$

expand along ***a row that's "uninvolved" with the ERO***

$$\det(EA) = g \det \begin{bmatrix} b & c \\ e+kb & f+kc \end{bmatrix} - h \det \begin{bmatrix} a & c \\ d+ka & f+kc \end{bmatrix} + i \det \begin{bmatrix} a & b \\ d+ka & e+kb \end{bmatrix}$$

why? because, without even evaluating the determinants, we know this row operation does not affect the determint for 2×2 matrices.

$$= g \det \begin{bmatrix} b & c \\ e & f \end{bmatrix} - h \det \begin{bmatrix} a & c \\ d & f \end{bmatrix} + i \det \begin{bmatrix} a & b \\ d & e \end{bmatrix}$$

$$= \det(A) \quad = 1 \cdot \det(A)$$

$$= \det(E) \cdot \det(A)$$

We then proved in class (see textbook):

If $E_1, \ldots, E_p$ are $n \times n$ elementary, and A, B are $n \times n$:

1) $\det(E_p \cdot \ldots \cdot E_1 A) = \det(E_p) \cdot \ldots \cdot \det(E_1) \cdot \det(A)$
2) $\det(E_p \cdot \ldots \cdot E_1) = \det(E_p) \cdot \ldots \cdot \det(E_1)$ (determinant of product = product of determints, for elementary matrices)
3) $\det(AB) = \det(A) \cdot \det(B)$ for any two $n \times n$ matrices A, B
and this easily generalizes to:
determinat of product = product of dweterminats for any number of square matrices.

(If you know 3), then 1) and 2) are of course automatically true; but we "inched up" to 3) by proving 1) and 2) first.

Theorem For an $n \times n$ matrix A, $\det A^T = \det A$

Proof True if A is 2×2 (*why?*

just check: is $det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = det\begin{bmatrix} a & c \\ b & d \end{bmatrix}$?

The proof is again "by induction": here's why "it works for 2×2" forces that "is works for 3×3." Why does this make it true when A is 3×3 ?

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad A^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$$

First row A = first column A^T; use this row & column to compute the determinants

det A=sum of $\downarrow$ (*across row 1)*	det A^T=sum of $\downarrow$ (*down column 1*)
$a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$	$a_{11}\begin{vmatrix} a_{22} & a_{32} \\ a_{23} & a_{33} \end{vmatrix}$
$-\,a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$	$-\,a_{12}\begin{vmatrix} a_{21} & a_{31} \\ a_{23} & a_{33} \end{vmatrix}$
$+\,a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$	$+\,a_{13}\begin{vmatrix} a_{21} & a_{31} \\ a_{22} & a_{32} \end{vmatrix}$

colored pairs have same determinant <u>because</u> they are 2×2 and transposes of each other

So $\det A = \det A^T$ when A is 3×3.

To show that, then $\det A = \det A^T$ must also be true for 4×4, then that it must also be true for 5×5, etc. uses exactly the same style calculation.