$n \times n$ **determinant** $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nj} & \cdots & a_{nn} \end{bmatrix}$

$A_{ij} = (n-1 \times (n-1)$ **matrix that remains after erasing i^{th} row and j^{th} column of A**

Example:

$$A_{1j} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nj} & \cdots & a_{nn} \end{bmatrix}$$

(going across the top row):

$$\begin{aligned} \det A = \quad & a_{11}(-1)^{1+1}\det(A_{11}) + \\ & a_{12}(-1)^{1+2}\det(A_{12}) + \ldots \\ & a_{1j}(-1)^{1+j}\det(A_{1j}) + \ldots \\ & a_{1n}(-1)^{1+n}\det A_{1n} \end{aligned}$$

Notation (to make stuff easier to write down clearly):

C_{ij} = (i,j) cofactor of A = the number = $(-1)^{i+j}\det(A_{ij})$

Definition of det A (going across first row of A)

$$\begin{aligned} \det A = \quad & a_{11}(-1)^{1+1}\det(A_{11}) + \\ & a_{12}(-1)^{1+2}\det(A_{12}) + \ldots \\ & a_{1j}(-1)^{1+j}\det(A_{1j}) + \ldots \\ & a_{1n}(-1)^{1+n}\det A_{1n} \end{aligned}$$

then becomes

$$\det A = a_{11}C_{11} + a_{12}C_{12} + \ldots + a_{1j}C_{1j} + \ldots a_{1n}C_{1n}$$

FACT: can compute det A using cofactor expansion going across any row or down any column – same answer!

across i^{th} row:

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \ldots + a_{in}C_{in}$$

$$(-1)^{i+1}\ \text{Det}A_{i1}$$

or

down j^{th} column:

$$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \ldots + a_{nj}C_{nj}$$

contains $(-1)^{1+j}$ $(-1)^{2+j}$ ...

Example $A = \begin{bmatrix} 1 & 1 & 3 \\ 0 & 2 & 0 \\ 1 & 2 & 1 \end{bmatrix}$ Find det A

Use cofactor expansion down the third column:

$a_{13}\, C_{13} + a_{23} C_{23} + a_{33} C_{33}$

$= a_{13}(-1)^{1+3} \det A_{13} + a_{23}(-1)^{2+3} \det A_{23} + a_{33}(-1)^{3+3} \det A_{33}$

$= 3 \det \begin{bmatrix} 0 & 2 \\ 1 & 2 \end{bmatrix} - 0 \det \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} + 1 \det \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$

$= 3(-2) + 0 + 1(2) = -4$

But easiest to choose a row or column with the most 0's (across 2nd row):

$$\det A = -0 \det \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix} + 2 \det \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix} - 0 \det \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = 2(-2) = -4$$

Example $\det A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 4 & 5 & 6 & 0 \\ 7 & 8 & 9 & 10 \end{bmatrix} = 1 \begin{bmatrix} 3 & 0 & 0 \\ 5 & 6 & 0 \\ 8 & 9 & 10 \end{bmatrix} = (1)(3) \det \begin{bmatrix} 6 & 0 \\ 9 & 10 \end{bmatrix}$

expanding along Row 1

$= (1)(3)(6) \det [10] = (1)(3)6)(10) = 180$. Determinant of a lower triangular matrix A is the product of its diagonal entries..

Example $\det B = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 8 & 9 \\ 0 & 0 & 0 & 10 \end{bmatrix} = (1) \det \begin{bmatrix} 5 & 6 & 7 \\ 0 & 8 & 9 \\ 0 & 0 & 10 \end{bmatrix} = (1)(5) \begin{bmatrix} 8 & 9 \\ 0 & 10 \end{bmatrix}$

expanding along Column 1

$= (1)(5)(8) \det [10] = (1)(5)(8)(10) = 400$. Determinant of an upper triangular matrix $\boldsymbol{B}$ is the product of its diagonal entries..

A square matrix U in an echelon form in an echelon is upper triangular:
so if $\det U$ is in echelon form, $\det U =$ the product of its diagonal elements.

In general, computing det A by expanding with cofactors along a row or a column is very inefficient and labor intensive unless A contains many 0's or has some special form such a upper triangular.

There is a more computationally efficient way to find det A :

if we use EROs to reduce A to an echelon form U, then

I) det is easy to compute, so
2) if we knew how each ERO affects the value of the determinant, we should be able to figure out det A once we calculate det U

EFFECT OF EROs ON DETERMINANTS

(*) Theorem Let A be a square matrix:

1) if a multiple of one row of A is added to another to get a matrix B, then det A = det B (*no effect on determinant*)

2) If two rows of A are interchanged to get B, then det B = $-$ det A
(*each interchange reverses sign on determinant*)

3) If one row of A is multiplied by k ($\neq 0$) to get B, then det $B = k$ det A
(*rescaling a row rescales the determinant too*)

$$\text{Example: } \det \overset{B}{\begin{bmatrix} 5a & 5b \\ c & d \end{bmatrix}} = 5 \det \overset{A}{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}$$

(3) is often used to "factor out" a number

Why is the Theorem true? For the next lecture.

Assuming these facts for now, calculate:

$$\det A = \det \begin{bmatrix} 2 & 2 & 0 \\ 1 & 1 & 3 \\ 0 & 2 & 1 \end{bmatrix} = -\det \begin{bmatrix} 1 & 1 & 3 \\ 2 & 2 & 0 \\ 0 & 2 & 1 \end{bmatrix} = -\det \begin{bmatrix} 1 & 1 & 3 \\ 0 & 0 & -6 \\ 0 & 2 & 1 \end{bmatrix}$$

$$= \ - \ - \det \begin{bmatrix} 1 & 1 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & -6 \end{bmatrix} = \det \begin{bmatrix} 1 & 1 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & -6 \end{bmatrix} = (1)(2)(-6) = -12$$

We used no row rescalings in the example (these are never necessary to get A into an echelon form) and two row interchanges (whose sign change effects therefore "cancelled" each other)

If A is square and A is row reduced to an echelon form U by a sequence of EROs with no row rescaling (*as can always be done!*), then

$$\begin{aligned} \det A &= (-1)^k \det U, \text{ where } k = \text{the number of row interchanges used.} \\ &= (-1)^k (\text{product of diagonal entries in } U) \end{aligned}$$

)

Therefore $\det A = 0$ if and only if there is at least one zero on the diagonal of U
if and only if U does not have n pivot positions
if any only if U does not have a pivot position in every column
if and only if A is not invertible.

Put a different way: If A is square, then

A is invertible if and only if there are no 0's are on the diagonal of U
if and only if $\det U \neq 0$
if and only if $\det A \neq 0$

Returning to the question of the effect of EROs on the determinant of A and looking at it from the point of view of elementary matrices.

Theorem If A is square $n \times n$ and
E is an $n \times n$ elementary matrix

then $\det(EA) = \det(E)\det(A)$

and

$$\det(E) = \begin{cases} 1 & \text{if } E \text{ is: add multiple of row to another} \\ -1 & \text{if } E \text{ is: a row interchange} \\ k & \text{if } E \text{ is: row rescaling by factor } k \end{cases}$$

Discuss this in next lecture